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LETTER TO THE EDITOR

Pressure and fractal indices for the Gibbs measures of hyperbolic Julia sets

G Servizi†, G Turchetti† and S Vaienti†‡

† Dipartimento di Fisica, via Irnerio 46, 40126 Bologna, Italy
‡ Centre de Physique Théorique Luminy, Laboratoire propre du CNRS, Case 907, F-13288 Marseille Cedex 09, France

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Abstract. We give a simple algorithm to compute the topological pressure for the Julia sets J of hyperbolic rational endomorphism. This allows us to compute the spectra of the generalised dimensions and entropies with respect to the whole class of Gibbs measures on J.

In recent literature some methods of computing the fractal properties of hyperbolic Julia sets have been given. These methods divide into two classes: in the first the various dimensions of the set are computed with covering procedures (box-counting) [1] or the escape rates are estimated toward different stable fixed points (final-state sensitivity) [2]. In the second class the Julia set is analysed by means of symbolic dynamics and the fractal indices arise from a thermodynamical description of the asymptotic distribution of the orbit on the invariant set [3-6]. In this letter we follow this second approach and show a simple but powerful technique to compute the fractal indices with respect to the uncountable set of the Gibbs measures. Our method seems to be much more simple, general and direct than the one proposed in [3] and is highly accurate. Before proceeding, we want to remark that all the following considerations apply not only to hyperbolic Julia sets (i.e. the critical points do not affect the set), but more generally to conformal mixing repellers (see [7] for the definitions and [8-10] for a few applications).

We denote with J the Julia set of the rational endomorphism T [11]; with $\|D_xT\|$ the norm of the tangent map of T at x and μ_σ a T-ergodic measure supported by J and indexed with $\sigma \in \boldsymbol{R}$. The starting point of our considerations is the existence, for any $\sigma \in \boldsymbol{R}$, of the limit [12]:

$$P(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y\in T^{-n}x} \frac{1}{\|D_yT^n\|^\sigma} = h(\mu_\sigma) - \sigma\lambda(\mu_\sigma). \tag{1}$$

x is any non-excluded [11] point of the complex plane; $h(\mu)$ and $\lambda(\mu)$ denote respectively the Kolmogorov entropy and the unique Lyapunov exponent of the measure μ, and μ_σ is the unique 'Gibbs' measure for which the equality in (1) holds [13]. $P(\sigma)$ is called the *pressure* of the function $-\sigma \log\|D_xT\|$ [13] and three values of σ have been particularly considered. When $\sigma = 0$, the corresponding measure μ_0 is the maximal entropy measure of T: $h(\mu_0)$ is the topological entropy [13]. When $\sigma = 1$ or 2, according to J extending on the line or in the plane, μ_σ is called the

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Sinai–Bowen–Ruelle measure, since the pressure equals the escape rate [8, 14]. Finally, when $\sigma = d_{\mathrm{H}}$ where d_{H} denotes the Hausdorff dimension of J, we have the well known Bowen–Ruelle formula: $P(D_{\mathrm{H}}) = h(\mu_{D_{\mathrm{H}}}) - D_{\mathrm{H}}\lambda(\mu_{D_{\mathrm{H}}}) = 0$ [7].

The pressure can also be expressed as the thermodynamical limit of a partition function, as [15, 8]:

$$P(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{A_\alpha^n \in \mathscr{A}^n} |A_\alpha^n|^\sigma \tag{2}$$

where $|A_\alpha^n|$ is the diameter of the αth atom $A_\alpha^n \in \mathscr{A}^n$, and $\mathscr{A}^n = \bigvee_{j=0}^{n} T^{-n}\mathscr{A}^0$, $\mathscr{A}^0$ being a generating closed partition of J. Now we choose a Markov partition [13] $\mathscr{A}^0$ of J (see [8] and [9] for the motivation of this choice) and introduce the dynamical free energy [16]:

$$F(q, \tau; \mu_\sigma) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{A_\alpha^n \in \mathscr{A}^n} \frac{\mu_\sigma(A_\alpha^n)^q}{|A_\alpha^n|^\tau}. \tag{3}$$

The generalised dimensions [9, 17] and the generalised Renyi entropies [18], can be defined in terms of the free energy. We stress the fact that they depend on the Gibbs measure on J. Before showing this, we relate the free energy to the pressure. The central step is the scaling of the Gibbs measure of an atom $A_\alpha^n \in \mathscr{A}^n$ [12]:

$$\mu_\sigma(TA_\alpha^n) = \int_{A_\alpha^n} \frac{\|D_x T\|^\sigma}{\exp(-P(\sigma))}\, \mathrm{d}\mu_\sigma(x). \tag{4}$$

Using this scaling in (3) and by asymptotic uniformity consideration we can prove that [5, 10]:

$$F(q, \tau; \mu_\sigma) = -qP(\sigma) + P(\sigma q - \tau). \tag{5}$$

The generalised dimensions with respect to μ_σ, $D_q(\mu_\sigma)$, are now defined by demanding that, for fixed q, the free energy must be zero. This uniquely defines a function τ_q^σ via:

$$P(\sigma q - \tau_q^\sigma) = qP(\sigma) \tag{6}$$

and the dimensions are:

$$D_q(\mu_\sigma)(q-1) = \tau_q^\sigma. \tag{7}$$

The generalised Renyi entropies $h_q(\mu_\sigma)$ are obtained putting $\tau = 0$ in (3) and dividing by $(1-q)$, i.e.

$$h_q(\mu_\sigma)(1-q) = F(q, 0; \mu_\sigma) = -qP(\sigma) + P(\sigma q). \tag{8}$$

The relations (7) and (8) have been recently obtained also in [19]. All the quantities given by (7) and (8) can be computed if the pressure is known and it can be estimated by means of the series (1). Another simple and accurate method to calculate the pressure only for disconnected repellers, has been proposed in [4].

In figure 1(a) and (b), we show respectively $D_q(\mu_\sigma)$ and $h_q(\mu_\sigma)$ against σ for the Julia set generated by the map $z' = z^2 - 0.15$ (J is connected).

In figure 2 we show the same quantities for the map $z' = z^2 - 3$ (J is a disconnected set on the real line). It could seem that the previous method could be applied to any hyperbolic rational endomorphism.

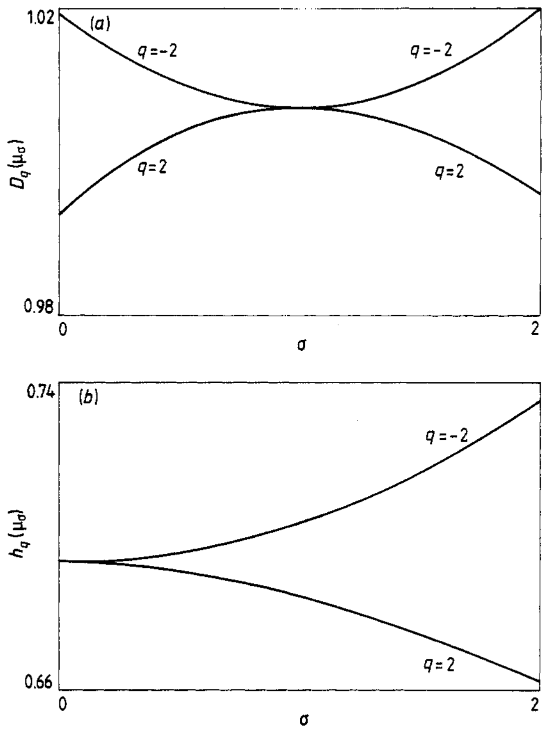


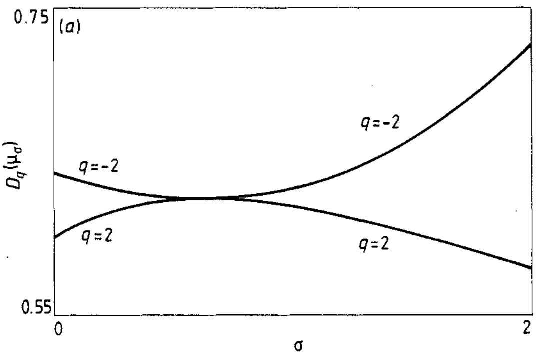
Figure 1. (a) For the map $z' = z^2 - 0.15$ the generalised dimensions $D_q(\mu_\sigma)$ corresponding to the Gibbs measure μ_σ are plotted against σ for two values of q. (b) The same for the generalised Renyi entropies $h_q(\mu_\sigma)$.

Although the convergence of (1) is true, and rather fast for the above examples, the rate of convergence of $(1/n)P_n(\sigma)$, where

$$P_n(\sigma) = \log \sum_{y \in T_z^{-n}} \|D_y T^n\|^{-\sigma}$$

is very slow and affected by exponentially decaying oscillations, when the Julia set is highly irregular. The oscillations in the pressure reflect in the computation of the generalised dimensions and this effect has been observed also in the calculation of the generalised dimensions for the Henon attractor using the box-counting technique [20]. These oscillations are related to the fact that the Julia set is strictly non-self-similar. In fact, let us consider a disconnected Julia set J and let us suppose that the Markov partition $\mathscr{A}^0$ of J consists of r rectangles, and put for simplicity diam $J = |J| = 1$. Then define the scales $\lambda^n_{\alpha,\beta} = |A^n_\alpha|/|A^{n-1}_\beta|$, which give the rate of dissection at the n-level, and suppose that the $\lambda^n_{\alpha,\beta}$ belong to the same set Λ for any $n > 0$. Thus $\Lambda = \{\lambda^0_i \mid i = 1, \ldots, r\}$ and any $|A^n_\alpha|$ is the product of a sequence of s^{n+1} elements belonging to Λ (s is the degree of the mapping T), and these sequences are in a one-to-one correspondence with the $|A^n_\alpha|$. We call strictly self-similar the invariant fractal which obeys this condition. But now it is easy to verify in (2) that for any finite n:

$$\frac{1}{n} \log \sum_{A^n_\alpha \in \mathscr{A}^n} |A^n_\alpha|^\sigma = \log \sum_{k=1}^{r} |\lambda^0_k|^\sigma = P(\sigma) \tag{9}$$



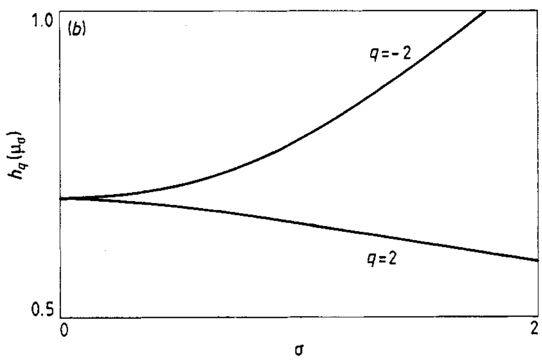


Figure 2. (a) For the map $z' = z^2 - 3$ the generalised dimensions $D_q(\mu_\sigma)$ corresponding to the Gibbs measure μ_σ are plotted against σ for two values of q. (b) The same for the generalised Renyi entropies $h_q(\mu_\sigma)$.

and since (2) is boundedly equivalent to (1), the oscillations in the pressure are absent. For example, this is the case for the piecewise linear expanding maps. If the rates of dissection of the scales are not uniform, the oscillations can appear.

In [5] we have extended the same techniques to the computation of the scaling function $f(\alpha)$ [17] and of the generalised Lyapunov exponents [9, 21].

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